

ON THE MEAN EXIT TIME FROM A MINIMAL SUBMANIFOLD

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Abstract

Let M^m be an immersed minimal submanifold of a Riemannian manifold N^n , and consider the Brownian motion on a regular ball $\Omega_R \subset M$ with exterior radius R . The mean exit time for the motion from a point $x \in \Omega_R$ is called $E_R(x)$. In this paper we find sharp support functions for $E_R(x)$ in the following distinct cases. The support is from below if the sectional curvatures of N are bounded from below by a nonnegative constant and the support is from above if the sectional curvatures of N are bounded from above by a nonpositive constant. It follows that the minimal submanifolds of \mathbf{R}^n all have the same mean exit time function and we show that this function actually characterizes the minimal hypersurfaces in the set of all hypersurfaces of \mathbf{R}^n .

1. Introduction

Let M^m be an immersed (not necessarily minimal) submanifold of a complete Riemannian manifold N^n . The distance function on N is denoted by r so that $\text{dist}_N(p, x) = r_p(x)$ for all p, x in N . We now fix $p \in M \subset N$ and define a *regular domain* $\Omega_R(p) \subset M$ to be a smooth connected component of $B_R(p) \cap M^m$ which contains p . Here $B_R(p)$ is the geodesic R -ball around p in N subject to the usual restriction that $R < \min\{i_N(p), \pi\sqrt{\kappa}\}$, where κ is the supremum of the sectional curvatures of N , and $i_N(p)$ is the injectivity radius of N from p .

We now consider the Brownian motion on the domain Ω_R (cf. [4]) and denote by $E_R(x)$ the mean time of first exit from Ω_R for a Brownian particle starting at $x \in \Omega_R$. We want to compare $E_R(x)$ with the mean exit time function $\tilde{E}_R^b(\tilde{x})$ on the space form R -ball $\tilde{B}_R^m(\tilde{p})$ of dimension $m = \dim M$ and constant curvature $b \in \mathbf{R}$. Since \tilde{B}_R has maximal isotropy at the center \tilde{p} , we have that \tilde{E}_R only depends on the distance of \tilde{x} from \tilde{p} . Hence we may, and do, write $\tilde{E}_R^b(\tilde{x}) = \tilde{\mathcal{E}}_R^b(r_{\tilde{p}}(\tilde{x})) = \tilde{\mathcal{E}}_R^b(r)$. In order to compare E and \tilde{E} we transplant \tilde{E} to Ω_R by the following definition:

$$\tilde{E}_R^b: \Omega_R \rightarrow \mathbf{R}; \quad \tilde{E}_R^b(x) = (\tilde{\mathcal{E}}_R^b \circ r_p)(x).$$

Our main result can now be formulated as follows.

Theorem 1. *Let M^m be a minimally immersed submanifold of N^n and let $\Omega_R(p)$ be a regular domain on M .*

(i) *If the sectional curvatures K_N of N satisfy $K_N \geq b \geq 0$, then $E_R(x) \geq \tilde{E}_R^b(x)$ for all $x \in \Omega_R$.*

(ii) *If $K_N \leq b \leq 0$, then $E_R(x) \leq \tilde{E}_R^b(x)$ for all $x \in \Omega_R$.*

Remark. This type of theorem was first proved by Debiard, Gaveau and Mazet in [3] where they used techniques different from ours to obtain bounds on the mean exit times from codimension-0 domains of N . The setting in Theorem 1 is more general in the sense that we have no a priori control on the sectional curvatures of the domains Ω_R in question.

A fundamental observation of Dynkin (cf. [4, vol. 2, p. 51]) states that the function $E_R(x)$ satisfies the Poisson equation on Ω_R with Dirichlet boundary data, i.e.,

$$(1.1) \quad \Delta_\Omega E_R = -1, \quad E_R|_{\partial\Omega_R} \equiv 0,$$

where Δ_Ω is the induced Laplacian on Ω_R defined by $\Delta_\Omega = \text{div}_\Omega \circ \text{grad}_\Omega = \text{trace}_\Omega \circ \text{Hess}_\Omega$.

If we let $G(x, y)$ denote the Green's functions with Dirichlet boundary conditions on Ω_R , then every smooth function f on Ω_R satisfies

$$f(x) = - \int_{\Omega_R} G(x, y) \Delta f(y) dy - \int_{\partial\Omega_R} \frac{\partial G}{\partial \xi}(x, y) f(y) dy.$$

In particular, if we set $f = E_R$ we get by (1.1)

$$E_R(x) = \int_{\Omega_R} G(x, y) dy.$$

On the other hand, the Green's functions are given by

$$G(x, y) = \int_0^\infty \mathcal{H}(x, y, t) dt,$$

where \mathcal{H} is the Dirichlet heat kernel on Ω_R . Thus

$$(1.2) \quad E_R(x) = \int_{\Omega_R} \int_0^\infty \mathcal{H}(x, y, t) dt dy,$$

and similarly

$$\tilde{E}_R^b(\tilde{x}) = \int_{\tilde{B}_R} \int_0^\infty \tilde{\mathcal{H}}_R^b(\tilde{x}, \tilde{y}, t) dt d\tilde{y}.$$

In [8] we showed that under the assumptions of Theorem 1(ii) we have

$$(1.3) \quad \mathcal{H}(p, y, t) \leq \tilde{\mathcal{H}}_R^b(r_p(y), t),$$

and also

$$(1.4) \quad \mathcal{H}(p, y, t) \leq \tilde{\mathcal{H}}_R^b(r_p(y), t),$$

where \mathcal{K} and $\tilde{\mathcal{K}}$ are the Neumann heat kernels on $\Omega_R(p)$ and $\tilde{B}_R(\tilde{p})$ respectively. From these latter kernels one may extract the volumes of the respective domains (cf. [2]) and obtain

$$(1.5) \quad \int_{\Omega_R} dy \geq \int_{\tilde{B}_R} d\tilde{y}.$$

It is now natural to ask for situations where inequalities of type (1.3) and (1.5) balance each other so as to give

$$(*) \quad E_R(x) = \tilde{E}_R^b(x) \quad \text{for all } x \in \Omega_R.$$

With this in mind we show the following gap phenomenon for minimal submanifolds of constant curvature space forms: The condition $(*)$ is a Bernstein type condition in every nonflat space form in the sense that Ω_R (and in fact all of M^m) must be totally geodesic if it is minimal and satisfies $(*)$. In contrast, $(*)$ is always satisfied in \mathbf{R}^n , where it even characterizes minimal hypersurfaces in the set of all hypersurfaces. To be precise we have the following theorems.

Theorem 2. *Under the assumptions of Theorem 1, suppose further that $(*)$ is fulfilled and that $b \neq 0$. Then Ω_R is a minimal cone in N^n and hence, if N^n is actually a space form of constant curvature b , then M^m is a totally geodesic submanifold of N^n .*

Theorem 3. *Let Ω_R be a regular domain of a minimal submanifold of \mathbf{R}^n . Then $(*)$ is satisfied. Conversely, if $(*)$ is satisfied for all x in a regular domain Ω_R of a hypersurface in \mathbf{R}^n , then Ω_R is minimal.*

We note here that Theorem 3 may be viewed as a semiglobal generalization of the following local result which is due to L. Karp and M. Pinsky (cf. [7]).

Theorem A. *Let M^{n-1} be a hypersurface of \mathbf{R}^n , and consider the domains $\Omega_\epsilon(p) \subset M$ for some fixed $p \in M$. If $E_\epsilon(p) = \tilde{E}_\epsilon^0(p)$ for all sufficiently small $\epsilon > 0$, then the mean curvature of M at p is zero.*

Remark. In order to construct the functional dependence of $E_\epsilon(p)$ on ϵ in this theorem it is in principle necessary to solve an infinite number of Poisson equations (one for every ϵ), whereas it is only necessary to solve a single Poisson equation (i.e., (1.1)) to check condition $(*)$ for Theorem 3.

2. Some preliminary inequalities

The following Hessian comparison theorem is well known and may be obtained from standard index comparison theory (cf. [1]).

Lemma 1. *If $K_N \leq b$ (respectively $K_N \geq b$), $b \in \mathbf{R}$, then for every unit vector X in the tangent bundle of $B_R(p) - p$ we have*

$$(2.1) \quad \text{Hess}_N(r_p)(X, X) \geq (\leq) h_b(r_p)(1 - \langle \text{grad}_N r_p, X \rangle^2),$$

where $h_b(r)$ is the constant mean curvature of any distance sphere of radius r in a space form of constant curvature b .

Now let F be a smooth function on \mathbf{R} . Then $F \circ r_p$ is a smooth function on $B_R(p) - p$ with a smooth restriction to $\Omega_R(p) - p$. A calculation along the lines of [6, p. 713] now gives

$$(2.2) \quad \begin{aligned} \text{Hess}_\Omega(F \circ r)(X, X) &= F''(r) \langle \text{grad}_N(r), X \rangle^2 \\ &\quad + F'(r) (\text{Hess}_N(r)(X, X) + \langle \text{grad}_N(r), \alpha(X, X) \rangle) \end{aligned}$$

for all $X \in T(\Omega_R(p) - p)$, where $\alpha(\cdot, \cdot)$ is the second fundamental form of Ω in N .

Combining (2.1) and (2.2) we get

Lemma 2. *Let $q \in \Omega_R(p) - p$ and let $\{X_i\}$, $1 \leq i \leq m$, be an orthonormal basis of $T_q\Omega \subset T_qN$. Suppose that $F'(r) \geq 0$ for all $r \in [0, R]$ and that $K_N \leq b$ (respectively $K_N \geq b$), $b \in \mathbf{R}$. Then*

$$(2.3) \quad \begin{aligned} \Delta_\Omega(F \circ r)|_q &\geq (\leq) (F''(r) - F'(r)h_b(r)) \cdot \sum_{j=1}^m \langle \text{grad}_N(r), X_j \rangle^2 \\ &\quad + mF'(r)(h_b(r) + \langle \text{grad}_N(r), H(q) \rangle), \end{aligned}$$

where $H(q)$ is the mean curvature vector of Ω at q in N .

In what follows we shall only consider functions $F = F_b$ defined by

$$(2.4) \quad F_b(r) = \begin{cases} \frac{1}{b}(1 - \cos(\sqrt{br})) & \text{if } b > 0, \\ r^2/2 & \text{if } b = 0, \\ \frac{1}{b}(1 - \cosh(\sqrt{-br})) & \text{if } b < 0. \end{cases}$$

Then $F_b \circ r$ is a smooth function on all of $\Omega_R(p)$, and moreover $F_b''(r) = F_b'(r)h_b(r)$ so that Lemma 2 specializes to

Corollary 3. *If $K_N \leq b$ (resp. $K_N \geq b$), then*

$$(2.5) \quad \Delta_\Omega(F_b \circ r)|_q \geq (\leq) mF_b'(r)(h_b(r) + pH_\Omega(q)),$$

where we have defined $pH_\Omega(q) = \langle \text{grad}_N(r), H(q) \rangle$, $pH_\Omega(p) = 0$.

3. Some properties of \tilde{E}

The mean time of first exit from a space form ball of constant curvature b , radius R and dimension m is given explicitly by (cf. [3])

$$(3.1) \quad \begin{aligned} \tilde{\mathcal{E}}_R^b(r) &= \int_r^R (\theta_b(u)u^{m-1})^{-1} \left(\int_0^u \theta_b(t)t^{m-1} dt \right) du \\ &= \tilde{\mathcal{E}}_R^b(0) - \tilde{\mathcal{E}}_r^b(0), \end{aligned}$$

where θ_b is the change of volume factor induced by the exponential map, i.e.,

$$(3.2) \quad \theta_b(r) = \begin{cases} (\sin \sqrt{br}/\sqrt{br})^{m-1} & \text{if } b > 0, \\ 1 & \text{if } b = 0, \\ (\sinh \sqrt{-br}/\sqrt{-br})^{m-1} & \text{if } b < 0. \end{cases}$$

Changing the variable from r to $s(r) = F_b(r)$ defined in (2.4) we get $\tilde{\mathcal{E}}_R^b(r) = \tilde{\mathcal{E}}_R^b(r(s)) = \mathbf{E}_R^b(s)$.

We shall need the following observations.

Proposition 4. $\mathbf{E}_R^b(s)$ is a smooth function of s for $0 \leq s \leq s(R)$, and in this interval we have

$$(3.3) \quad \frac{d}{ds} \mathbf{E}_R^b(s) < 0 \quad \text{for every } b,$$

$$(3.4) \quad \frac{d^2}{ds^2} \mathbf{E}_R^b(s) \begin{cases} < 0 & \text{if } b > 0, \\ = 0 & \text{if } b = 0, \\ > 0 & \text{if } b < 0. \end{cases}$$

Proof. Since $r(s)$ and $\tilde{\mathcal{E}}_R^b(r)$ (in (3.1)) are smooth when the respective arguments are nonzero, we see that $\mathbf{E}(s)$ is a smooth function of s for $0 < s \leq s(R)$. The fact that $\tilde{E}_R^b(x)$ solves the Poisson equation (1.1) on $\tilde{B}_R(\tilde{p})$ can therefore be written as

$$(3.5) \quad -1 = \tilde{\Delta} \tilde{E} = \mathbf{E}''(s) \cdot \|\text{grad}_{\tilde{B}} s\|^2 + \mathbf{E}'(s) \cdot \tilde{\Delta} s,$$

and so from (2.5),

$$-1 = \mathbf{E}''(s) \cdot (s'(r))^2 + \mathbf{E}'(s) \cdot m s'(r) h_b(r),$$

or equivalently

$$(3.6) \quad -1 = \mathbf{E}''(s) \cdot s(2 - bs) + \mathbf{E}'(s) \cdot m(1 - bs).$$

The unique solution $\mathbf{E}(s)$ to (3.6) with $\mathbf{E}(0) = c_0$ has a power series expansion $\mathbf{E}(s) = \sum_{k=0}^{\infty} c_k s^k$ whose radius of absolute convergence is $2/|b|$. Indeed, a substitution of the series into (3.6) leads to the recurrence relation

$$c_1 = -1/m, \quad c_{k+1} = c_k \cdot b(k^2 + (m-1)k)/(2k^2 + (m+2)k + m), \quad k \geq 1,$$

and the claim now follows from the ratio test together with

$$|c_{k+1}|/|c_k| \rightarrow |b|/2 \quad \text{for } k \rightarrow \infty.$$

In particular $\mathbf{E}_R^b(s)$ is therefore smooth in the closed interval $0 \leq s \leq s(R)$, and

$$(3.7) \quad \begin{aligned} & \mathbf{E}_R^b(s) - \mathbf{E}_R^b(0) \\ &= -\frac{1}{m}s - \frac{b}{2(m+2)}s^2 - \frac{b^2 \cdot (m+1)}{3(m+2)(m+4)}s^3 - \dots, \quad s < \frac{2}{|b|}. \end{aligned}$$

We see that $\mathbf{E}'(0) = -1/m < 0$, and the inequality (3.3) follows for all other values of s from $r'(s) > 0$ and $\tilde{\mathcal{E}}'(r) < 0$ (from (3.1)). Furthermore, the series (3.7) also gives directly the statement (3.4) for $b > 0$ and $b = 0$ respectively. In case $b < 0$ we first observe from (3.7) that $\mathbf{E}''(0) = -b/(m+2) > 0$. A differentiation of (3.6) with respect to s gives

$$0 = (\mathbf{E}''(s))' \cdot s(2 - bs) + (\mathbf{E}''(s))(m+2)(1 - bs) - mb\mathbf{E}'(s).$$

Here $mb\mathbf{E}'(s)$ is positive so that $\mathbf{E}''(s) = \eta(s)$ satisfies

$$0 < \eta'(s) \cdot s(2 - bs) + \eta(s) \cdot (m+2)(1 - bs).$$

If a first zero s_0 exists for η , then $\eta'(s_0) > 0$, but this is clearly ruled out by $\eta(0) > 0$. Hence $\mathbf{E}''(s)$ is positive for all s when $b < 0$.

4. Proof of Theorems 1 and 2

The comparison of $E_R(x)$ with $\mathbf{E}_R^b(s(r_p(x)))$ will follow from the identity

$$(4.1) \quad \Delta_\Omega \mathbf{E}_R^b(s(r(x))) = \mathbf{E}''(s) \cdot \|\text{grad}_\Omega s\|^2 + \mathbf{E}'(s) \cdot \Delta_\Omega s.$$

In fact, suppose that $K_N \geq b > 0$. Then by Corollary 3 (with $H_\Omega = 0$), Proposition 4 and the fact that $\|\text{grad}_\Omega s\|^2 \leq \|\text{grad}_N s\|^2 = \|\text{grad}_{\tilde{B}_R} s\|^2$ we get

$$\begin{aligned} \Delta_\Omega \tilde{E}_R^b(x) &\geq \mathbf{E}''(s) \cdot \|\text{grad}_N s\|^2 + \mathbf{E}'(s) \cdot ms'(r)h_b(r) \\ &\geq \mathbf{E}''(s) \cdot \|\text{grad}_{\tilde{B}_R} s\|^2 + \mathbf{E}'(s) \cdot \tilde{\Delta}_{\tilde{B}_R} s \\ &= \tilde{\Delta}_{\tilde{B}_R} \tilde{E}_R^b = -1 = \Delta_\Omega E_R(x). \end{aligned}$$

Therefore $\tilde{E}_R^b(x) - E_R(x)$ is a subharmonic function on Ω_R vanishing on $\partial\Omega_R$ so that the maximum principle applies and gives $\tilde{E}_R^b(x) \leq E_R(x)$ for all $x \in \Omega_R$.

In case $K_N \leq b < 0$ we get similarly

$$\Delta_\Omega \tilde{E}_R^b(x) \leq \tilde{\Delta}_{\tilde{B}_R} \tilde{E}_R^b = -1 = \Delta_\Omega E_R(x).$$

Thus $\tilde{E}_R^b(x) - E_R(x)$ is a superharmonic function vanishing on the boundary so that $\tilde{E}_R^b(x) \geq E_R(x)$ for all $x \in \Omega_R$. This proves Theorem 1.

If $K_N \equiv b = 0$, then by continuity $\tilde{E}_R^b(x) = E_R(x)$ for all $x \in \Omega_R$. Conversely, if $\tilde{E}_R^b(x) = E_R(x)$ for all $x \in \Omega_R$, and we are in one of the two cases of Theorem 1 with $b \neq 0$, then we conclude from $(\mathbf{E}_R^b(s))'' \neq 0$ that $\|\text{grad}_\Omega s\| = \|\text{grad}_N s\|$. Hence Ω_R is a minimal cone in N . If N has constant curvature $K_N \equiv b \neq 0$, then by analytic continuation from $\Omega_R = \tilde{B}_R$ we finally get that all of M^m is a totally geodesic submanifold of N^n . q.e.d.

We note here that if $\Omega_R(p)$ is a codimension-0 domain of N we get $\|\text{grad}_\Omega s\| = \|\text{grad}_N s\|$ for free, so that the assumption of $E_R(x) = \tilde{E}_R^b(x)$ for all $x \in \Omega_R(p)$ now gives (from (4.1))

$$(4.2) \quad \Delta_\Omega s = \tilde{\Delta}_{\tilde{B}_R} s \quad \text{for all } s.$$

Under any of the curvature assumptions $\text{Ric}_N \geq (n-1)b$ or $K_N \leq b$, (4.2) is only possible if $K_N \equiv b$ along every 'radial' plane in $\Omega_R(p)$, which therefore must be isometric to the space form ball \tilde{B}_R of constant curvature b (cf. [1]). Thus we get the following rigidity theorem which completes the Corollaire in [3, p. 796].

Proposition 5. *Let $\Omega_R^n(p)$ be a regular domain of N^n . Suppose that either $\text{Ric}_N \geq (n-1)b$ or $K_N \leq b$ for some $b \in \mathbf{R}$, and that $E_R(x) = \tilde{E}_R^b(x)$ for all $x \in \Omega_R$. Then Ω_R is isometric to \tilde{B}_R with constant curvature b .*

5. Submanifolds of \mathbf{R}^n

Now suppose again that Ω_R is a regular domain of any submanifold of \mathbf{R}^n . Then

$$(5.1) \quad \begin{aligned} \Delta_\Omega \tilde{E}_R^0(x) &= \mathbf{E}'(s) \cdot \Delta_\Omega s \\ &= \mathbf{E}'(s) \cdot ms'(r)(h_0(r) + pH_\Omega(x)) \\ &= \tilde{\Delta}_{\tilde{B}_R} \tilde{E}_R^0(x) + \mathbf{E}'(s) \cdot mr(x)pH_\Omega(x) \\ &= \Delta_\Omega E_R(x) + mr(x)\mathbf{E}'(s) \cdot pH_\Omega(x). \end{aligned}$$

If we assume that $E_R(x) = \tilde{E}_R^0(x)$ for all $x \in \Omega_R$, then we must have $pH_\Omega(x) \equiv 0$.

To establish Theorem 3 stated in §1 we therefore only have to prove the following

Lemma 6. *Let M^{n-1} be an immersed hypersurface of \mathbf{R}^n . If M has everywhere vanishing pH , then M is minimal.*

Proof. Suppose for contradiction that $H(q) \neq 0$ for some $q \in M$. Then $H \neq 0$ in some maximal neighborhood $\mathcal{U}(q) \subset M$. Since

$$pH(x) = \langle \text{grad}_{\mathbf{R}^n} r, H \rangle = 0,$$

and H is orthogonal to the tangent space $T_x \mathcal{U}$, we get $\text{grad}_{\mathbf{R}^n}(x) \in T_x \mathcal{U}$. Thus $Z(x) = \text{grad}_{\mathbf{R}^n} r(x)$ is a unit vector field whose straight line integral curves foliate \mathcal{U} . Hence \mathcal{U} is part of a cone with vertex p . But on a cone in \mathbf{R}^n the length of the mean curvature vector grows to ∞ as one approaches the vertex. In particular $\mathcal{U}(q)$ must contain p where $\|H\| = \infty$. This contradicts the smoothness of M and proves the lemma.

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